# A Space of Semi-Magic Squares

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An interesting paper appears in the May, 1999 issue of the *American Mathematical Monhtly.* "Marriage, Magic, and Solitaire" by David Leep and Gerry Myerson covers a range of points, starting with talking about how we can prove that the game of Solitaire can always be won. But the title is somewhat of a misnomer, the core of the paper talks about how the set of semi-magic squares (those whose rows and columns all add up to the same number) forms a vector space, and how we can prove / understand the size of these spaces. This concept is interesting mainly due to its arbitrariness, how is it that such an arbitrary group such as semi-magic squares could form a vector space?

Before launching into the mathematics, it is probably educational to discuss solitaire, the ostensible "problem" that all of this math is designed to solve. The game of solitaire is not actually one game (despite what the proliferation of the Windows version might have us believe), but is a family of games with many many variations (according to wikipedia). Thus, one can think of solitaire like poker, there are many variants of the game but they all share some common elements. The common elements of all solitaire games are that they are played with a game of 52 playing cards, only 1 player, and the player places the cards down in front of him in a number of piles. The goal of the game is to choose the right card from each pile (usually of cards of the same suit).

Now the version that Leep and Myerson consider is when the player arranges the 52 cards in 13 columns, 4 cards in each column. The point of this version is to select one card from each column, so that you have all 13 denominations (two through ace) in your hand at the end. The question is: can one always win the game?

## Marriage

One answer comes from Hall's marriage principle. The principle states that:

$$There\ exists\ distinct\ x_1, ..., x_n\ such\ that\ x_j \in A_j\ for\ all\ j,\ if\ and\ only\ if$$
$$Size(\cup_{j \in J}) \geq Size(J)\ for\ all\ J \subset \{1, ..., n\}.$$

In other words, if we have n objects, the x's, and we have j sets, the A's, then each set has at least one of the objects if and only if we can choose some arbitrary J < n, and the union of the sets A1, ..., AJ has at least J elements.

As a baseline example of this, think of if we have 4 objects and 3 sets. The marriage principle states that each of the sets contains one of these 4 objects if and only if the union of the first 2 of these sets has at least 2 members, and the union of the first 3 of these sets has at least 3 members.

Why is it called the marriage principle? Because when it was first applied, the "objects" were good young men, and the "sets" were the set of men that a given woman could marry. The point was to prove that there was one good man for each woman out there (another interesting problem... though not in the same way as solitaire).

What does this all mean for solitaire? One can think of the sets as each of the 13 columns and the objects as the 13 denominations (whether they be ace, king, two, etc.). Is there at least one of each object in each set? Yes, because any collection of k columns must have k different numbers of cards (because each column has 4 cards, making 4k cards total, and since each denomination only has 4 suits, there is at least k different denominations in there). This satisfies the conditions of the marriage principle (any union of k sets has at least k different denominations in them).

Okay, so that was pretty cool, yes? As another example of the marriage principle, we could go back and ask the question: how do we know there is a good many that every woman can marry?

Well, suppose we have 10 men, and 10 women in the world, each with their own number. Each woman has only dated the corresponding man with her number, and the next one. So woman #4 has dated men #4 and #5, woman #2 has dated men #2 and #3, and so on. And finally, suppose that each woman will only date a man she has dated. How do we know that each woman will find a man to marry?

Using the marriage principle, we can see that yes, clearly there is. We know this because the union of any k sets of dated men, even if they are for women of adjacent numbers, will contain at least k+1 different men.

**A Space of Squares**

Another proof that solitaire is solvable also involves the marriage principle, and the vast majority of the paper, involves the concept of semi-magic squares, or squares whose columns and rows add up to the same number (known as the magic number of the square). An example of a semi-magic square with a magic number of 7 appears below:

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

Now by the marriage principle, we know that there exists a transversal of the matrix consisting of non-zero entries. A transversal being a selection of elements, one from each row, and no two in the same column.

Let us think of the game of solitaire as a 13x13 matrix. Each column representing one of the 13 columns in the game. Each row in the matrix representing one of the 13 denominations (two through ace). Now, if there always exists a transversal of the matrix consisting of non-zero elements (as Hall's principle says we should), then clearly the game of solitaire is always winnable.

But is there anything else interesting that the paper tells us? Well yes, in Theorem 3 the authors state that:

Every semi-magic square can be expressed as a sum of permutation matrices.

A permutation matrix being a matrix that represents a permutation, or a matrix with only 1's and 0's, no two 1's being in the same row or the same column. And example of the permutation matrix representing (14)(23) is presented below.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Note, that all permutation matrices are, themselves, semi-magic squares.

Now authors claim that these semi-magic squares are a vector space. What is their basis? Well they are the set of all permutation matrices (including the identity permutation) that are the identity or transpositions or three-cycles that involve 1 (so any permutation of the form (1a) or (1ab)). As an example, let us take the magic square we looked at earlier.

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

How can we write this as a sum of transpositions and three-cycles? See the example below.

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix} = 1 * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 2 * \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$+ 2 * \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Well what about the dimension of this vector space? What is that? Well, the authors give many ways to calculate this, but perhaps the easiest way is just to find the size of the basis.

Let us work with semi-magic squares of size nxn. Recall that the basis consists of all permutation matrices that represent permutations of the form: (1a), (1ab), or the identity.

How many permutations are of the form (1a)? Clearly (1-n) because you only have n-1 choices for a (a cannot be 1 or we have the identity).

How many permutations are of the form (1ab)? Well we can choose any of the (n-1) remaining elements to swap with 1, and then any of the remaining (n-2) elements to put in b such that 1, a, and b are unique.

How many permutations are equal to the identity? Clearly, just 1 (the identity itself).

So the dimension of our space is of size: (n-2)(n-1) + (n-1) + 1 = n^2 – 2n +2.

The rest of the paper then goes on to talk about various properties of our vector space, many of which are too long to talk about in the remaining space here. But the important and interesting part of the paper is still the proof of the existence of such a vector space. Would would have thought that you could construct a vector space by just taking the identity matrix, the transposition matrices, and the three-cycle matrices, and that this vector space would not only consist of semi-magic squares, but that it would have a direct application to the proof of solitaire as an always-winnable game.