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Mathematics 152
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8 December 2008

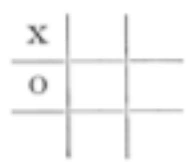
Reading Project: The Group of Automorphisms of the Game of Three-Dimensional Tick Tack Toe

The group of automorphisms of the game of 3-d tick tack toe is a very interesting topic and is relevant to our study of automorphisms in class. By studying the automorphisms of the 3-d tick tack toe game, and other games that involve symmetric grids, we can understand which moves are essentially identical and prune the “move tree by eliminating positions equivalent to the one already considered.” This kind of study is important in analyzing different strategies to play the game or for developing a computer program that will be playing the game.

2-d Case

To understand the automorphisms of the tick tack toe game we can use the 2-d case with which we are all very familiar. In the following example we have an X in the top left corner and an O right below it.

We can think of an automorphism as a mapping of the board that preserves three in a row, or preserves the collinearity of three points on the board. The automorphisms of the board are a subgroup of the

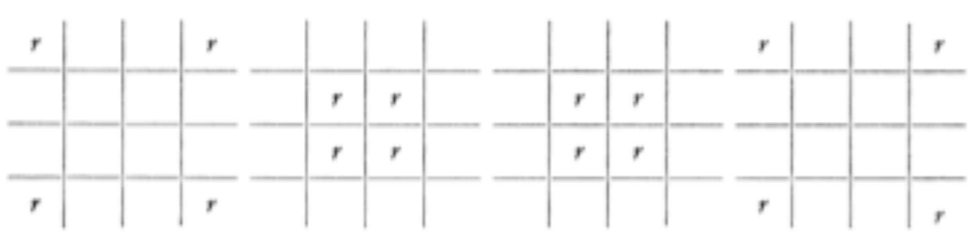


general group of mappings of the board which has order 9!. The automorphisms of this 3 x 3 board are simply its rotations and reflections. By rotating and reflecting the above board, we can find 7 different arrangements that are essentially equivalent.

3-d Case

Now let us consider the case of 64 unit cubes in the larger 4 x 4 x 4 3-d cube. In this game, you win if you can mark 4 unit cubes in a row or you tie if the entire grid is filled but neither player has marked 4 unit cubes in a row.

In order to more easily see the markings on the larger cube, we can break the 64 unit cubes into 4 cross sections or 4, 4 x 4 grids and refer to each unit cube by using 3-d grid coordinates.



For example, the cubes in the first grid, which is the bottom cross section, can be represented by the coordinates (x, y, 0) where 0 <= x,y <= 3.

We can classify the possible points into 2 different categories and the possible lines into 4 different categories.

·Rich points: A point is rich if the coordinates (x, y, z) are either all 0 or 3 or the coordinates are all 1 or 2. The rich points are designated in the above diagram.
·Poor points: A point is poor if it is not rich.
·Full line: A full line has 4 rich points.
·Empty Line: An empty line has 4 poor points.
·Diagonal Line: A diagonal line has 2 rich and 2 poor points, and passes through 2 opposite edges of the cube.
·Ranking Line: A ranking line has 2 rich and 2 poor points, and is parallel to a coordinate axis.

One important property to note is that all the sets above are invariant, or they do not change under a set of transformations. This property will be important later.

<u>The Subgroup of Automorphisms of the 3-d Case</u>

Again we can consider the group of mappings of the cube onto itself, which has an order 64!. We can call this group T_3 and we can call the subgroup of automorphisms A_3. What are the automorphisms in the case of the cube? The automorphisms preserve the collinearity of four points, or preserve lines of four.

·The group R of the rotations of the cube: From our studies in class we know that there are 24 symmetries or rotations of the cube that will map it back onto itself. We also know that these rotations will preserve lines so each one of the rotations is an automorphism.
·The Evisceration Mapping E: (x, y, z) ? (xe, ye, ze) where 0e=1, 1e=0, 2e=3, 3e=2
·This essentially turns the board inside out, maps the interior rich points onto the exterior rich points, and is an automorphism.
·The Scramble Mapping S: (x, y, z) ? (xs, ys, zs) where 0s=0, 1s=2, 2s=1, 3s=3
·It maps each interior rich point to its opposite interior rich point and is an automorphism.
·
What is most interesting is that we can use the group R of automorphisms, the Evisceration Mapping, and the Scramble Mapping to construct the entire 192 member subgroup A_3. The proof is actually quite lengthy and a bit tedious but I think it is important to highlight a couple of important points. Using E and S we can construct an 8 member subgroup of A_3 that we can call F.

$$\begin{aligned} e &= (01)(23) & e^2 &= i \\ s &= (12) & s^2 &= i \\ es &= (0231) \\ se &= (0132) \\ ese &= (03) \\ ses &= (03)(12) \\ eses = sese &= (03)(12). \end{aligned}$$

Then we can construct the entire subgroup A_3 by taking the direct product of R and F. Because R has 24 elements and F has 8 elements, R x F has 8 x 24 = 192 elements. To prove that this direct product makes up the entire subgroup we need to show two things.

1Given an automorphism A, there is an X in R x F such that AX leaves the four points
2(0, 0, 0), (1, 0, 0), (3, 0, 0), and (0, 3, 0) fixed.
3
4If an automorphism A leaves the four points above fixed, then A = I.
5
I will not go into the proof, but these conditions are satisfied and the direct product R x F forms the entire subgroup of automorphisms, A_3.

Important Implications

What are some of the important implications of an automorphism on our invariant sets of points and lines?

1If A fixes 3 points on a line, then it also fixes the fourth point.
2If A leaves a diagonal or ranking line invariant, and fixes one of its rich points, then it fixes the other rich point.
3Let p be a rich point, and let p_1, p_2, p_3 be the rich points on diagonal or ranking lines through p. If A fixes p_1, p_2, p_3, then A fixes P.

Conclusions

The great thing about this analysis is that it can be extended to other games that use a grid system in order to understand the symmetries of the board and to understand what moves would be identical according to the rules of the game. I really enjoyed the article and would consider extending this analysis myself.